# COMMENTS

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## Comment on "Negative temperature of vortex motion"

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In a recent Brief Report and subsequently [Phys. Rev. A **43**, 2050 (1991); **44**, 8439 (1991)], Berdichevsky, Kunin, and Hussain claim that the "Boltzmann temperature" of a bounded point vortex system is always positive, and that the spatial inhomogeneities that evolve at high energies in such a system are incompatible with ergodicity of the dynamics. The argument given to support these claims neglected the presence of the fluid boundary. We prove that the Boltzmann temperature is in fact always *negative*, and present evidence that the vortex clumping that has been observed in simulations is consistent with ergodic dynamics.

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In a Brief Report [1], Berdichevsky, Kunin, and Hussain state that the "paradoxical" situation of negative temperature states for bounded vortex systems can be resolved by using a definition for the entropy of the microcanonical ensemble that is more appropriate than the usual one for systems with finitely many particles. Specifically, the entropy $S(E)$ of the system at energy $E$ is taken to be $\log\Gamma(E)$, where $\Gamma(E)$ is the volume of the region in state space of states having energy not greater than $E$. Since $\Gamma$ is an increasing function, the thermodynamic definition of temperature $1/T \equiv dS/dE$ shows that the temperature is positive at all energies. Furthermore it is claimed that the temperature as just defined is identical to the "Boltzmann temperature" $\langle x_i \partial H/\partial x_i \rangle$. Here $H$ is the energy function, $x_i$ is any coordinate in Euclidian phase space, and the angular brackets denote the integral with respect to the canonical invariant measure on the set of states with energy $E$. Both definitions of temperature are valid for systems with finitely many, even few, degrees of freedom. In a subsequent Reply [2] to a Comment on their original Brief Report, Berdichevsky *et al.* write, "The appearance of ordered structures like clouds of positive and negative vortices is evidence of the *nonergodicity* of motion rather than a manifestation of negative temperature."

In this Comment we wish to point out that for a bounded vortex system, the Boltzmann temperature is in fact *negative* for all energies. Furthermore we argue that vortex clumping is *not* inconsistent with ergodic dynamics, and in fact the usual methods of statistical mechanics (including the assumption of ergodicity) predict an inhomogeneous equilibrium state which is in agreement with simulations.

In a previous work [3] Berdichevsky established that, if each level surface of the energy function $H$ is assumed to be compact and to bound the region in phase space of states having equal or lesser energy, then the following relation holds:

$$\int_{H(\mathbf{x})=E} x_i \frac{\partial H}{\partial x_i} \frac{dA}{\|\nabla H\|} = \Gamma(E) . \tag{1}$$

The integral is over the states with energy $E$ and $dA$ is the volume element of this hypersurface. When properly normalized, this integral gives the Boltzmann temperature; the normalization constant is simply $1/\Gamma'(E)$.

The assumptions underlying (1) do not hold for bounded vortex systems because the boundary is "attractive," that is, $H \to -\infty$ as any vortex tends to the boundary. Thus the boundary of the set of states having energy not greater than $E$ consists of the level set $H(\mathbf{x})=E$ and the boundary of state space itself. (Recall that in a vortex system there is no momentum, and the state space for a system of $N$ particles in a region $D$ is merely $D^N$.) For example, for a single vortex in a rectangular region, the set of states with $H<E$ consists of the region between the curve $H=E$ and the rectangular boundary.

The effect of this additional boundary is to add the constant $-\Gamma(\infty)$ to the right-hand side of (1). A simple way to see this is to consider the same system with a different energy function, namely, $\tilde{H}=-H$. The boundary is now "repulsive," and (1) is valid, with $H$, $E$, and $\Gamma(E)$ replaced by $\tilde{H}$, $\tilde{E}=-E$, and $\tilde{\Gamma}(\tilde{E})=\Gamma(\infty)-\Gamma(E)$, respectively. [The proof of (1) as given in Ref. [3] must be modified slightly, because of the singularity in $H$ when two particle positions coincide.] Thus for bounded vortex systems one finds that the Boltzmann temperature is

$$\left\langle x_i \frac{\partial H}{\partial x_i} \right\rangle = \frac{\Gamma(E) - \Gamma(\infty)}{\Gamma'(E)} . \tag{2}$$

This quantity is negative for all $E$.

As pointed out in Ref. [1], the sign of the Boltzmann temperature of a vortex system has geometric significance if the dynamics are ergodic. Negative values correspond to an average counterclockwise motion of the positive vortices in the region and clockwise motion of the negative ones. This result is consistent with a considerable body of evidence that has appeared over the past twenty years (again see the citations in Ref. [1]) which has shown that neutral vortex systems, at sufficiently high energies, tend to evolve to a state in which all the positive vortices are moving counterclockwise in a single eddy, and the negative vortices clockwise in another. Furthermore, a simulation [4] of a small neutral vortex system (with periodic boundary conditions) shows that at very low energies, vortices tend to join up in neutral pairs and move rapidly about more or less independently of one another, except for scattering exchanges. In the presence of a boundary, a large negative average value of $\langle x_i \partial H / \partial x_i \rangle$ would result.

The expected value in (2) is an average over states. If the dynamics of the system are ergodic, this corresponds (with probability 1) to the time average along a trajectory in state space. In Ref. [2] the claim is made that the emergence of vortex clusters is evidence for the nonergodicity of vortex dynamics, i.e., the evolution of ordered structures is inconsistent with ergodicity. Ergodicity of vortex dynamics is an open question, although there is some evidence that vortex dynamics are not ergodic for a system with only six vortices [4]. Nevertheless, numerical work [5] done on neutral vortex systems with periodic boundary conditions shows that the emergence of coherent structures is consistent with ergodicity. In this work, state space averages were taken of a certain function which is a natural diagnostic of vortex clustering. The results clearly show that states with two vortex clusters are the "most probable" states at high energies. Any ergodic dynamics for this system would exhibit an evolution in time towards this most probable state. This behavior is seen, at least qualitatively, in simulations of continuous vorticity distributions [6].

Recently Eyink and Spohn [7] have studied the microcanonical ensemble of regularized point vortices in the "mean-field" limit, where the number of vortices goes to infinity while the region and the total vorticity remain constant. (This differs from the usual thermodynamic limit, and is the appropriate limit to take when discussing models of continuous fluid flow.) They find a unique macroscopic equilibrium state (vorticity distribution) for energies above a threshold. Moreover they find the *same* distribution using the canonical ensemble, *provided the temperature is negative.* In the context of the canonical ensemble it seems clear that negative temperature is not just a synonym for high energy but rather has a physical significance, i.e., vortices of the same sign "statistically attract."

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